Basic Calculus: Differentiation

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1. First Principle of Differentiation

Notes

The derivative of a function $f(x)$ from first principles is defined as:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Examples

Example 1: Differentiate $f(x) = x^2$ using the first principle.

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{2xh + h^2}{h} = \lim_{h \to 0} (2x + h) = 2x$$

Example 2: Differentiate $f(x) = x^3 + 3x$ using the first principle.

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^3 + 3(x+h) - (x^3 + 3x)}{h} = \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3 + 3h}{h} = \lim_{h \to 0} (3x^2 + 3xh + h^2 + 3) = 3x^2 + 3$$

Worksheet

1. Differentiate $f(x) = x^3$ using the first principle.
2. Differentiate $f(x) = 2x^2 + 4x$ using the first principle.
3. Differentiate $f(x) = x^4$ using the first principle.
4. Differentiate $f(x) = x^2 - x$ using the first principle.
5. Differentiate $f(x) = 5x^3$ using the first principle.
6. Differentiate $f(x) = 6x^2 + 2x$ using the first principle.
7. Differentiate $f(x) = 2x^4 - x^2$ using the first principle.
8. Differentiate $f(x) = x^5 - 3x$ using the first principle.
9. Differentiate $f(x) = 3x^2 + 2x^3$ using the first principle.
10. Differentiate $f(x) = x^6$ using the first principle.

Solutions

1. $f'(x) = 3x^2$
2. $f'(x) = 4x + 4$
3. $f'(x) = 4x^3$
4. $f'(x) = 2x - 1$
5. $f'(x) = 15x^2$
6. $f'(x) = 12x + 2$
7. $f'(x) = 8x^3 - 2x$
8. $f'(x) = 5x^4 - 3$
9. $f'(x) = 6x + 6x^2$
10. $f'(x) = 6x^5$

2. Product Rule

Notes

The product rule states that if $u(x)$ and $v(x)$ are functions of x, then:

$$\frac{d}{dx}[u(x)v(x)] = u'(x)v(x) + u(x)v'(x)$$

Examples

Example 1: Differentiate $f(x) = x^2 \cdot (3x + 1)$.

$$f'(x) = \frac{d}{dx}[x^2] \cdot (3x + 1) + x^2 \cdot \frac{d}{dx}[3x + 1] = 2x \cdot (3x + 1) + x^2 \cdot 3 = 6x^2 + 2x + 3x^2 = 9x^2 + 2x$$

Example 2: Differentiate $f(x) = (x^3 - x)(2x^2 + 3)$.

$$f'(x) = \frac{d}{dx}[x^3 - x] \cdot (2x^2 + 3) + (x^3 - x) \cdot \frac{d}{dx}[2x^2 + 3] = (3x^2 - 1) \cdot (2x^2 + 3) + (x^3 - x) \cdot 4x$$
$$= 6x^4 + 9x^2 - 2x^2 - 3 + 4x^4 - 4x^2 = 10x^4 - 5x^2 - 3$$

Worksheet

1. Differentiate $f(x) = x \cdot (4x^2 + 1)$ using the product rule.
2. Differentiate $f(x) = (x^2 + 2) \cdot (x^3 - 1)$ using the product rule.
3. Differentiate $f(x) = (x^2 - 3x) \cdot (2x + 5)$ using the product rule.
4. Differentiate $f(x) = (3x^2) \cdot (x^2 + 4)$ using the product rule.
5. Differentiate $f(x) = (x + 1) \cdot (x^4 - x^2)$ using the product rule.
6. Differentiate $f(x) = (2x - 1) \cdot (x^3 + 3x)$ using the product rule.
7. Differentiate $f(x) = (x^3 + 4) \cdot (x^2 - 2)$ using the product rule.
8. Differentiate $f(x) = (5x - 2) \cdot (x^3 + 1)$ using the product rule.
9. Differentiate $f(x) = (x^2 + x) \cdot (3x^2 - x)$ using the product rule.
10. Differentiate $f(x) = (4x + 3) \cdot (2x^2 - x)$ using the product rule.

Solutions

1. $f'(x) = 12x^2 + 1$
2. $f'(x) = 5x^4 + 6x^2 - 2x$
3. $f'(x) = 6x^2 - 2x - 15$
4. $f'(x) = 12x^3 + 24x$
5. $f'(x) = 5x^4 + 4x^3 - 3x^2 - 3$
6. $f'(x) = 8x^3 - 3x^2 + 12x - 3$
7. $f'(x) = 5x^4 - 6x^2 + 8x$
8. $f'(x) = 20x^3 - 3x^2 + 5$
9. $f'(x) = 12x^3 + 6x^2 - 2x$
10. $f'(x) = 24x^2 + 8x - 7$

3. Quotient Rule

Notes

The quotient rule states that if $u(x)$ and $v(x)$ are functions of x, then:

$$\frac{d}{dx}\left[\frac{u(x)}{v(x)}\right] = \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}$$

Examples

Example 1: Differentiate $f(x) = \frac{x^2}{x+1}$.

$$f'(x) = \frac{(2x)(x+1) - (x^2)(1)}{(x+1)^2} = \frac{2x^2 + 2x - x^2}{(x+1)^2} = \frac{x^2 + 2x}{(x+1)^2}$$

Example 2: Differentiate $f(x) = \frac{x^3-1}{x^2+2}$.

$$f'(x) = \frac{(3x^2)(x^2+2) - (x^3-1)(2x)}{(x^2+2)^2} = \frac{3x^4 + 6x^2 - 2x^4 + 2x}{(x^2+2)^2} = \frac{x^4 + 6x^2 + 2x}{(x^2+2)^2}$$

Worksheet

1. Differentiate $f(x) = \frac{x^2+3}{x}$ using the quotient rule.
2. Differentiate $f(x) = \frac{2x^3}{x^2+1}$ using the quotient rule.
3. Differentiate $f(x) = \frac{x^4-1}{x+2}$ using the quotient rule.
4. Differentiate $f(x) = \frac{x^3+4}{x^2-1}$ using the quotient rule.
5. Differentiate $f(x) = \frac{3x^2-2x}{x^3+1}$ using the quotient rule.
6. Differentiate $f(x) = \frac{2x+1}{x^2+x}$ using the quotient rule.
7. Differentiate $f(x) = \frac{x^2+2x}{x^3}$ using the quotient rule.
8. Differentiate $f(x) = \frac{x^2-3x}{x^2+3x}$ using the quotient rule.
9. Differentiate $f(x) = \frac{x^4+x}{x^2+1}$ using the quotient rule.
10. Differentiate $f(x) = \frac{5x^3}{x^2-x}$ using the quotient rule.

Solutions

1. $f'(x) = \frac{(2x)(x)-(x^2+3)}{x^2} = \frac{x^2-3}{x^2}$
2. $f'(x) = \frac{6x^3(x^2+1)-2x^3(2x)}{(x^2+1)^2} = \frac{2x^4+6x^2}{(x^2+1)^2}$
3. $f'(x) = \frac{4x^3(x+2)-(x^4-1)}{(x+2)^2} = \frac{3x^4+8x^3+1}{(x+2)^2}$
4. $f'(x) = \frac{(3x^2)(x^2-1)-(x^3+4)(2x)}{(x^2-1)^2} = \frac{x^4-3x^2-8x}{(x^2-1)^2}$
5. $f'(x) = \frac{(6x-2)(x^3+1)-(3x^2-2x)(3x^2)}{(x^3+1)^2} = \frac{6x^4-6x-2x^3-2-9x^4+6x^3}{(x^3+1)^2} = \frac{-3x^4+4x^3+6x-2}{(x^3+1)^2}$
6. $f'(x) = \frac{(2)(x^2+x)-(2x+1)(2x+1)}{(x^2+x)^2} = \frac{-2x^2-2x-1}{(x^2+x)^2}$
7. $f'(x) = \frac{(2x+2)(x^3)-(x^2+2x)(3x^2)}{x^6} = \frac{2x^4+2x^3-3x^4-6x^3}{x^6} = \frac{-x^4-4x^3}{x^6}$
8. $f'(x) = \frac{(2x-3)(x^2+3x)-(x^2-3x)(2x+3)}{(x^2+3x)^2} = \frac{2x^3+6x^2-3x^2-9x-2x^3+6x^2-3x^2+9x}{(x^2+3x)^2} = \frac{6x^2}{(x^2+3x)^2}$
9. $f'(x) = \frac{(4x^3+1)(x^2+1)-(x^4+x)(2x)}{(x^2+1)^2} = \frac{4x^5+4x^3+x^2+1-2x^5-2x^3}{(x^2+1)^2} = \frac{2x^5+2x^3+x^2+1}{(x^2+1)^2}$
10. $f'(x) = \frac{(15x^2)(x^2-x)-(5x^3)(2x-1)}{(x^2-x)^2} = \frac{15x^4-15x^3-10x^4+5x^3}{(x^2-x)^2} = \frac{5x^4-10x^3}{(x^2-x)^2}$

4. Power Rule

Notes

The power rule states that if $f(x) = x^n$, then:

$$f'(x) = nx^{n-1}$$

Examples

Example 1: Differentiate $f(x) = x^5$.

$$f'(x) = 5x^4$$

Example 2: Differentiate $f(x) = x^{\frac{3}{2}}$.

$$f'(x) = \frac{3}{2}x^{\frac{1}{2}}$$

Worksheet

1. Differentiate $f(x) = x^4$ using the power rule.
2. Differentiate $f(x) = x^3$ using the power rule.
3. Differentiate $f(x) = x^2$ using the power rule.
4. Differentiate $f(x) = x^{\frac{1}{2}}$ using the power rule.
5. Differentiate $f(x) = x^{\frac{3}{4}}$ using the power rule.
6. Differentiate $f(x) = x^6$ using the power rule.
7. Differentiate $f(x) = x^{-\frac{2}{3}}$ using the power rule.
8. Differentiate $f(x) = x^7$ using the power rule.
9. Differentiate $f(x) = x^{\frac{5}{2}}$ using the power rule.
10. Differentiate $f(x) = x^{-\frac{3}{4}}$ using the power rule.

Solutions

1. $f'(x) = 4x^3$
2. $f'(x) = 3x^2$
3. $f'(x) = 2x$
4. $f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$
5. $f'(x) = \frac{3}{4}x^{-\frac{1}{4}}$
6. $f'(x) = 6x^5$
7. $f'(x) = -\frac{2}{3}x^{-\frac{5}{3}}$
8. $f'(x) = 7x^6$
9. $f'(x) = \frac{5}{2}x^{\frac{3}{2}}$
10. $f'(x) = -\frac{3}{4}x^{-\frac{7}{4}}$

5. Chain Rule

Notes

The chain rule states that if $f(x) = g(h(x))$, then:

$$\frac{d}{dx}[f(x)] = g'(h(x)) \cdot h'(x)$$

Examples

Example 1: Differentiate $f(x) = (x^2 + 1)^3$.

$$f'(x) = 3(x^2 + 1)^2 \cdot \frac{d}{dx}[x^2 + 1] = 3(x^2 + 1)^2 \cdot 2x = 6x(x^2 + 1)^2$$

Example 2: Differentiate $f(x) = (2x^3 + x)^4$.

$$f'(x) = 4(2x^3 + x)^3 \cdot \frac{d}{dx}[2x^3 + x] = 4(2x^3 + x)^3 \cdot (6x^2 + 1) = 4(2x^3 + x)^3 \cdot (6x^2 + 1)$$

Worksheet

1. Differentiate $f(x) = (x^3 + 2)^2$ using the chain rule.
2. Differentiate $f(x) = (x^2 - 1)^5$ using the chain rule.
3. Differentiate $f(x) = (3x^2 + 4x)^3$ using the chain rule.
4. Differentiate $f(x) = (x^4 - 2)^3$ using the chain rule.
5. Differentiate $\sqrt{x^2 + x + 2}$ using the chain rule.
6. Differentiate $f(x) = (x^2 + x + 1)^4$ using the chain rule.
7. Differentiate $f(x) = (2x^2 - 3x)^3$ using the chain rule.
8. Differentiate $f(x) = (4x^3 + 5)^{20}$ using the chain rule.
9. Differentiate $f(x) = (x^5 - 1)^3$ using the chain rule.
10. Differentiate $f(x) = (x^2 + 3x)^4$ using the chain rule.

Solutions

1. $f'(x) = 2(x^3+2)(3x^2) = 6x^2(x^3+2) = 6x^5 + 12x^2$
2. $f'(x) = 5(x^2-1)^4 \cdot (2x) = 10x(x^2-1)^4$
3. $f'(x) = 3(3x^2+4x)^2 \cdot (6x+4) = (18x+12)(3x^2+4x)^2$
4. $f'(x) = 3(x^4-2)^2 \cdot (4x^3) = 12x^3(x^4-2)^2$
5. $f(x) = (x^2+x+2)^{1/2} \implies f'(x) = \frac{1}{2}(x^2+x+2)^{-\frac{1}{2}}(2x+1) = \frac{2x+1}{2(x^2+x+2)^{\frac{1}{2}}} = \frac{2x+1}{2\sqrt{x^2+x+2}}$
6. $f'(x) = 4(x^2+x+1)^3 \cdot (2x+1) = (8x+4)(x^2+x+1)^3$
7. $f'(x) = 3(2x^2-3x)^2 \cdot (4x-3) = (12x-9)(2x^2-3x)^2$
8. $f'(x) = 20(4x^3+5)^{19} \cdot (12x^2) = 240x(4x^3+5)^{19}$
9. $f'(x) = 3(x^5-1)^2 \cdot 5x^4 = 15x^4(x^5-1)^2$
10. $f'(x) = 4(x^2+3x)^3 \cdot (2x+3) = (8x+12)(x^2+3x)^3$